ISRAEL JOURNAL OF MATHEMATICS, Vol. 72, No. 3, 1990

# SKEW FIELDS OF DIFFERENTIAL OPERATORS

BY
ABRAHAM A. KLEIN[a],† AND LEONID MAKAR-LIMANOV[b],‡
[a]*Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel; and*
[b]*Department of Mathematics, Wayne State University, Detroit, MI 48202, USA*

ABSTRACT
We show in this note that two curves are defined up to birational equivalence by the skew fields of differential operators on these curves.

The main result of this note is the following:

THEOREM. *Let $F$ be a field of characteristic 0 and $K_1, K_2$ field extensions of $F$ of transcendental degree 1. Let $\delta_1, \delta_2$ be derivations of $K_1, K_2$, respectively, each with subfield of constants $F$. Consider the skew fields $D_1 = K_1(x_1; \delta_1)$, $D_2 = K_2(x_2; \delta_2)$. Then $D_1$ and $D_2$ are $F$-isomorphic if and only if $K_1$ and $K_2$ are $F$-isomorphic.*

Let $A$ be a commutative algebra. It is possible to attach in a natural way a non-commutative algebra $\mathfrak{D}(A)$ [4] to such an algebra, the so-called ring of differential operators on $A$. If one starts with an irreducible algebraic curve $\Gamma$ over a field $F$ of zero characteristic and considers $\mathfrak{D}(\Gamma) = \mathfrak{D}(\mathcal{O}(\Gamma))$, where $\mathcal{O}(\Gamma)$ is the coordinate ring of $\Gamma$, then $\mathfrak{D}(\Gamma)$ is called the ring of differential operators on $\Gamma$ [9]. The algebras $\mathfrak{D}(\Gamma)$ satisfy the Ore condition, so for any $\Gamma$ there is a uniquely determined skew field $\mathrm{Frac}(\mathfrak{D}(\Gamma))$ which is the skew field of quotients of $\mathfrak{D}(\Gamma)$. These skew fields are of the type considered in the theorem. Indeed, let $K = K(\Gamma)$ be the field of rational functions on $\Gamma$ (i.e., the field of fractions of $\mathcal{O}(\Gamma)$). Then $\mathfrak{D}(\Gamma) \subset \mathfrak{D}(K)$ [7, 15.5.5(iii)] and any $f \in \mathfrak{D}(K)$ may be represented as $\sum_{i=0}^{n} f_i \delta^i$ where $f_i \in K$, and $\delta$ is any nontrivial $F$-derivation of $K$ [7, 15.2.5, 15.5.5(ii)]. Therefore

†The project was started when this author was visiting Wayne State University.
‡This author is supported by an NSF grant.
Received January 1, 1990

$\mathfrak{D}(K)$ is isomorphic to $K[x;\delta]$. Furthermore, $\mathrm{Frac}(\mathfrak{D}(\Gamma))$ contains $K$, and since it contains a nontrivial derivation [7, 15.3.6], which we may assume to be $\delta$, $\mathrm{Frac}(\mathfrak{D}(\Gamma)) \simeq \mathrm{Frac}(\mathfrak{D}(K)) \simeq K(x;\delta)$. So in this setting, the result of the theorem means that the curve $\Gamma$ is defined up to birational equivalence by the skew field $\mathrm{Frac}(\mathfrak{D}(\Gamma))$.

We start with one field extension $K$ of $F$. Let $\delta \neq 0$ be a derivation of $K$ with subfield of constants $F$, then each $y \in K \backslash F$ is transcendental over $F$. Consider the ring of differential polynomials $K[x;\delta]$ and its skew field of quotients $D = K(x;\delta)$. One easily shows that the center of $D$ is $F$ since $F$ has zero characteristic (e.g., by making use of the representation of the elements of $D$ given later). If $y \in K\backslash F$ then $D = K(x';\delta')$ where $x' = \delta(y)^{-1}x$ and $\delta' = \delta(y)^{-1}\delta$. If $\mathrm{tr.deg}\, K/F = 1$ then $\delta'$ is the unique extension of $d/dy$ on $F(y)$, since $K$ is algebraic over $F(y)$. Using these remarks we get the proof of the easy direction of the theorem, namely, if $K_1$ and $K_2$ are $F$-isomorphic then $D_1$ and $D_2$ are $F$-isomorphic. Take $y_1 \in K_1\backslash F$ and let $y_2$ be its image under a given $F$-isomorphism $\sigma: K_1 \to K_2$, then $\sigma(F(y_1)) = F(y_2)$. Let $x_i' = \delta_i(y_i)^{-1}x_i$ and $\delta_i' = \delta_i(y_i)^{-1}\delta_i$, $i = 1,2$. It follows that if $a_1 \in K_1$, $a_2 \in K_2$ and $\sigma(a_1) = a_2$ then $\sigma(\delta_1'(a_1)) = \delta_2'(a_2)$ and we get an $F$-isomorphism $K_1[x_1';\delta_1'] \simeq K_2[x_2';\delta_2']$ which extends to $D_1 \simeq D_2$.

The other direction of the theorem is much more difficult and holds with no restriction on the transcendental degree. We split the proof into several lemmas, each being interesting by itself. We first introduce the notations and definitions that will be used.

The elements of $D = K(x;\delta)$ will be written as a Laurent series in $x^{-1}$, namely, $d = \sum_{i \leq n} a_i x^i$, $a_i \in K$ [2, p. 18]. If $a_n \neq 0$ we let $\deg d = n$ and, as usual, $\deg 0 = -\infty$. We call $a_n x^n$ the *leading term* of $d$, which we denote by $|d|$, and we set $|0| = 0$. The coefficient of the leading term of $d$, namely $a_n$, will be denoted by $c(d)$. Note that $c(d_1 d_2) = c(d_1)c(d_2)$.

For $f \in D$, let $R_f^k = \{g \in D \mid ad_f^{k+1} g = 0\}$, $k = 0,1,\ldots$ . Then $R_f^0 = C(f)$ is the centralizer of $f$ in $D$; $R_f^0 \subset R_f^1 \subset R_f^2 \subset \cdots$ and $R_f = \bigcup R_f^k$ is a subring of $D$. Let $(R_f) = \{gh^{-1} \mid g,\, h \in R_f,\, h \neq 0\}$. We say that $f$ is *ad-nilpotent* if $(R_f) = D$. For example, if $a \in K$ then $R_a^0 \supseteq K$ and $R_a^1 \ni x$. It follows that $R_a \supseteq K[x;\delta]$ and $(R_a) = D$. Thus, if $N(D)$ denotes the set of ad-nilpotent elements of $D$, then $N(D) \supseteq K$.

We define the "*Poisson bracket*" $\{f,g\}$ of two elements $f, g \in D$. If one of them is 0 we set $\{f,g\} = 0$. If $f \neq 0$ *and* $g \neq 0$, let $|f| = f_n x^n$, $|g| = g_m x^m$ and define

$$\{f,g\} = (nf_n\delta(g_m) - mg_m\delta(f_n))x^{n+m-1}. \tag{1}$$

As usual, we denote $fg - gf$ by $[f,g]$. Note that $\deg[f,g] < \deg f + \deg g$ for any $0 \neq f,\, g \in D$. If $\deg[f,g] = \deg f + \deg g - 1$ then $\{f,g\} = |[f,g]|$ and otherwise

$\{f,g\}=0$. In particular, $\{f,g\}=0$ if $[f,g]=0$. We also introduce the notation $\mathrm{Ad}_f g$ for $\{f,g\}$ and observe that $\mathrm{Ad}_f$ is not linear, but it acts on products almost like a derivation. We have:

$$\mathrm{Ad}_f(gh) = |(\mathrm{Ad}_f g)h| + |g(\mathrm{Ad}_f h)| \tag{2}$$

and we easily get, for any integer $n$,

$$\mathrm{Ad}_f(g^n) = n|g^{n-1}\mathrm{Ad}_f g|. \tag{3}$$

Our first result generalizes a result of Amitsur [1, Th. 1(2)].

LEMMA 1. *If $f$ is a noncentral element of $D$, then its centralizer $C(f)$ is commutative.*

PROOF. $C(f)$ is clearly a skew field and we assume it is not commutative. Let $a$ be a noncentral element of $C(f)$. We claim that we may assume $\deg a \neq 0$. For if $\deg a = 0$ for any noncentral element $a$ of $C(f)$, then if $a_1, a_2 \in C(f)$ we have either $[a_1,a_2]=0$, or if $[a_1,a_2]\neq 0$ then, since $\deg[a_1,a_2] < \deg a_1 + \deg a_2 = 0$, $[a_1,a_2]$ is central in $C(f)$.

Now we take a noncentral element $a$ of $C(f)$ with $\deg a \neq 0$ and define for $0 \neq d \in C(f)$: $\mathrm{def}(d) = \deg[a,d] - \deg d$. It is clear that there exists $d \in C(f)$ such that $\mathrm{def}(d) \neq -\infty$ and def is bounded above by $\deg a$. Let $b \in C(f)$ be an element for which $\mathrm{def}(b)$ is maximal, then of course $[a,b]\neq 0$.

Let $|f| = f_n x^n$, $|a| = a_r x^r$, $|b| = b_s x^s$. We have $r \neq 0$ and we show that we may assume $n \neq 0$. Since $[f,a]=0$, we have $nf_n\delta(a_r) - ra_r\delta(f_n) = 0$, and if $n = 0$ then $ra_r\delta(f_0)=0$ and $\delta(f_0)=0$. So $f_0 \in F$ and $f$ may be replaced by $g = f - f_0$, which has degree $\neq 0$ and the same centralizer.

Thus, we have $n \neq 0$ and $nf_n\delta(a_r) = ra_r\delta(f_n)$. Now $[f,b]=0$ implies $nf_n\delta(b_s) = sb_s\delta(f_n)$, so we get $sb_s\delta(a_r) = ra_r\delta(b_s)$. This implies $\delta(a_r^{-s}b_s^r)=0$ so $b_s^r = \alpha a_r^s$ for some $\alpha \in F$. It follows that $\deg(b^r - \alpha a^s) < \deg b^r$ and if $b^r - \alpha a^s \neq 0$ then $\mathrm{def}(b^r - \alpha a^s) > \mathrm{def}(b^r)$. Now if we can show that $\mathrm{def}(b^r) = \mathrm{def}(b)$, then in particular $[a,b^r]\neq 0$ so $b^r - \alpha a^s \neq 0$, and we will have a contradiction to the maximality of $\mathrm{def}(b)$. In fact, we claim that $\mathrm{def}(d^m) = \mathrm{def}(d)$ for any $0 \neq d \in C(f)$ and any integer $m \neq 0$. It suffices to prove this for $m > 0$ and $m = -1$. The proof is obtained by making use of $\deg[a,d^m] = (m-1)\deg d + \deg[a,d]$, which is a result of the identities $[a,d^m] = \sum_{i=0}^{m-1} d^i[a,d]d^{m-1-i}$ for $m>0$ and $[a,d^{-1}] = -d^{-1}[a,d]d^{-1}$ for $m=-1$.

Our next result shows that, usually, an ad-nilpotent element has zero degree.

LEMMA 2. *If $N(D)$ contains an element of nonzero degree, then $K$ is a simple transcendental extension of $F$.*

PROOF. Let $f \in N(D)$ and $|f| = f_n x^n$ with $n \neq 0$. We first show that $\{f, g\} \neq 0$ for some $g \in R_f$. Assume $\{f, g\} = 0$ for each $g \in R_f$. If $0 \neq d \in D$ then, since $(R_f) = D$, there exist $0 \neq g, h \in R_f$ with $d = gh^{-1}$. By assumption, $\{f, g\} = 0$ and $\{f, h\} = 0$. Let $|g| = g_m x^m$, $|h| = h_{m'} x^{m'}$, then $nf_n \delta(g_m) = mg_m \delta(f_n)$ so $g_m^n = \alpha f_n^m$ for some $\alpha \in F$ and, similarly, $h_{m'}^n = \beta f_n^{m'}$, $\beta \in F$. It follows that $c(d)^n = (g_m h_{m'}^{-1})^n = \alpha\beta^{-1} f_n^{m-m'}$. In particular, if $d \in K \backslash F$ then $m = m'$ and $d^n = \gamma$ for some $\gamma \in F$, but this contradicts the fact that $d$ is transcendental over $F$.

Now choose $g \in R_f$ for which $\{f, g\} \neq 0$. Let us extend the field $K$ by an element $u$ satisfying $u^n = f_n$ and let $L = K[u]$. Denote the unique extension of $\delta$ to $L$ by the same letter $\delta$. We get an extension of $D$, namely $L(x; \delta)$, in which we perform the following calculations. We have $0 \neq \{f, g\} = \{f_n x^n, g\} = \{u^n x^n, g\} = \{(ux)^n, g\}$. Since $g \in R_f$ we have $\mathrm{ad}_f^m g = 0$, $m \geq 2$. It follows that $\mathrm{Ad}_{(ux)^n}^m g = 0$. By (3) we have $\{(ux)^n, g\} = n|(ux)^{n-1}\{ux, g\}|$ so $\{ux, g\} \neq 0$ and, applying $\mathrm{Ad}_{(ux)^n}$ $m$ times, we get $\mathrm{Ad}_{(ux)^n}^m g = n^m |(ux)^{m(n-1)} \mathrm{Ad}_{ux}^m g|$, so $\mathrm{Ad}_{ux}^m g = 0$. Let $k$ be such that $\mathrm{Ad}_{ux}^k g \neq 0$ and $\mathrm{Ad}_{ux}^{k+1} g = 0$, then $k \geq 1$ since $\{ux, g\} \neq 0$. Let $\mathrm{Ad}_{ux}^{k-1} g = a$ and $\{ux, a\} = b$; then $\{ux, b\} = 0$ and by (2) we get $\{ux, b^{-1}a\} = 1$. If we denote $b^{-1}a$ by $v$ we have $\{ux, v\} = 1$ and $v \in L$.

Next we prove that if $ax^m$ is the leading term of an element of $R_f$ then $a = u^m p(v)$ for some polynomial $p(v) \in F[v]$. Given $g \in R_f$ we get, as above, that $\mathrm{Ad}_{ux}^{k+1} g = 0$ for some $k \geq 0$, so if $|g| = ax^m$ we have $\mathrm{Ad}_{ux}^{k+1}(ax^m) = 0$. Now we prove that $a = u^m p(v)$, $p(v) \in F[v]$, just from the assumption $\mathrm{Ad}_{ux}^{k+1}(ax^m) = 0$. We argue by induction. If $k = 0$ then $\{ux, ax^m\} = 0$ so $\delta(au^{-m}) = 0$ and $a = u^m \alpha$, $\alpha \in F$. Thus, in this case, we take $p(v) = \alpha$. To proceed from $k - 1$ to $k$, let $\{ux, ax^m\} = bx^m$; then $\mathrm{Ad}_{ux}^k(bx^m) = 0$, so $bx^m = |(ux)^m p(v)|$ for some $p(v) \in F[v]$. Let $q(v) \in F[v]$ with $q'(v) = p(v)$; then $\{ux, q(v)\} = q'(v)\{ux, v\} = p(v)$ and $\{ux, (ux)^m q(v)\} = |(ux)^m \{ux, q(v)\}| = |(ux)^m p(v)| = bx^m = \{ux, ax^m\}$. Since $(ux)^m q(v)$ and $ax^m$ have the same degree, it follows that $\{ux, ax^m - (ux)^m q(v)\} = 0$ unless $\deg(ax^m - (ux)^m q(v)) < m$, in which case $a = u^m q(v)$. If this is not the case, then we have the case $k = 0$ for $|ax^m - (ux)^m q(v)|$ so $a - u^m q(v) = u^m \alpha$, $\alpha \in F$, but $q(v) + \alpha \in F[v]$ as required.

We are ready to finish the proof of the lemma. Let $d \in D = (R_f)$, then $d = gh^{-1}$, with $g, h \in R_f$. By what we have just proved, $|g| = ax^r = u^r p_1(v) x^r$, $|h| = bx^s = u^s p_2(v) x^s$ with $p_1(v), p_2(v) \in F[v]$. So $|d| = u^{r-s} p(v) x^{r-s}$ with $p(v) = p_1(v) p_2(v)^{-1} \in F(v)$. In particular, for $d \in K$ we get $d = |d| \in F(v)$, so $K \subseteq F(v)$ and therefore, by Lüroth's theorem [6, p. 515], $K$ is a simple transcendental extension of $F$.

A more precise result than that of the previous lemma is:

COROLLARY. *$N(D)$ contains elements of nonzero degree if and only if $D$ is the skew field of quotients of the Weyl algebra $A_1(F)$.*

PROOF. If $N(D)$ contains elements of nonzero degree, then by the above lemma $K = F(y)$ and, as we have seen, we may assume $\delta = d/dy$. Then $F[y][x;\delta]$ is generated over $F$ by $x, y$ and $[x,y] = 1$, so $F[y][x;\delta] = A_1(F)$. Now it is clear that $D = K(x;\delta)$ is the skew field of quotients of $A_1(F)$. On the other hand, this skew field contains $x$, which is of degree 1, and in $N(D)$ since $R_x^k \ni y^k$, so $R_x \supseteq A_1(F)$ and $(R_x) = D$.

Another interesting result is the following:

LEMMA 3. *For any ad-nilpotent element $f$ not in $F$, the skew field $D$ is the field of quotients of a ring of differential polynomials over the centralizer of $f$.*

PROOF. Let $E = C(f)$. Since $(R_f) = D$, $R_f$ is not commutative and let $g \in R_f \backslash E$. Thus, for some $k \geq 1$, we have $\mathrm{ad}_f^k g \neq 0$, $\mathrm{ad}_f^{k+1} g = 0$ and, as in the previous lemma for $a = \mathrm{ad}_f^{k-1} g$, $b = [f,a] \neq 0$, we get $[f, b^{-1}a] = 1$. So if $t = -b^{-1}a$ then $[t,f] = 1$ and $t \in R_f$. Since $E \subseteq R_f$ we get $E[t] \subseteq R_f$. Let us prove the reverse inclusion. We have $R_f^0 = E$ and assume $R_f^{i-1} \subseteq \sum_{j=0}^{i-1} Et^j$. Let $h \in R_f^i$, then $[h,f] \in R_f^{i-1}$ so $[h,f] = \sum_{j=0}^{i-1} a_j t^j$, $a_j \in E$. If

$$v = \sum_{j=0}^{i-1} \frac{a_j}{j+1} t^{j+1}$$

then $[v,f] = \sum_{j=0}^{i-1} a_j t^j$ since $[t,f] = 1$. So $[h - v, f] = 0$, implying that $h - v \in E$ and $h \in \sum_{j=0}^{i} Et^j$.

If $a \in E$ then $[[t,a],f] = [[t,f],a] + [t,[a,f]] = 0$ so $[t,a] = a' \in E$, and it follows that the map $a \to a'$ is a derivation $\delta'$ of $E$. The subfield of constants is $F$ since $\delta'(a) = 0$ for $a \in F$ and if $a \in E \backslash F$ then, as in [3, Cor. 4.3], $C(a) = C(f)$ so $\delta'(a) = [t,a] \neq 0$ since $[t,f] \neq 0$. We have proved above that $R_f = \sum Et^j$ so $R_f = E[t;\delta']$ and, since $f \in N(D)$, we get $D = E(t;\delta')$.

COROLLARY. *If $K$ is a simple transcendental extension of $F$ and $f \in N(D) \backslash F$ then $C(f)$ is $F$ isomorphic to $K$.*

PROOF. Again let $E = C(f)$, so $E$ is a field which contains $F$ and, by the previous lemma, $D = E(t;\delta')$. We have started with $D = K(x;\delta)$, and we may assume $K = F(y)$ and $[x,y] = 1$. Thus $[x,y]$ has degree 0 and so either $x$ or $y$ has $t$-degree $\neq 0$. But $x, y \in N(D)$ so $N(D)$ has an element of $t$-degree $\neq 0$. It follows by Lemma 2 that $E$ is a simple transcendental extension of $F$ and therefore $E$ and $K$ are $F$-isomorphic.

LEMMA 4. *If $f \in N(D) \backslash F$ then $C(f) \subseteq N(D)$.*

PROOF. We prove that if $f, g \in D \backslash F$ commute then $R_f = R_g$ so, in particular, if $f \in N(D) \backslash F$ and $g \in C(f)$ then $g \in N(D)$. By symmetry it suffices to prove $R_f \subseteq R_g$. Since $C(f)$ is commutative and $g \in C(f)$ we have $R_f^0 = C(f) \subseteq C(g) = R_g^0$. Assume $R_f^{k-1} \subseteq R_g^{k-1}$ and let $a \in R_f^k$, so $\mathrm{ad}_f a \in R_f^{k-1} \subseteq R_g^{k-1}$, implying $\mathrm{ad}_g^k(\mathrm{ad}_f a) = 0$. But $\mathrm{ad}_f$ and $\mathrm{ad}_g$ commute, so $\mathrm{ad}_f(\mathrm{ad}_g^k a) = 0$, implying $\mathrm{ad}_g^k a \in C(f) \subseteq C(g)$ and therefore $a \in R_g^k$.

Lemma 5. *If $K$ is not a simple transcendental extension of $F$ and $f \in N(D) \setminus F$, then $C(f)$ is $F$-isomorphic to $K$.*

Proof. Here too we denote $C(f)$ by $E$. Thus $E$ is a subfield contained in $N(D)$. By Lemma 2, each of the nonzero elements of $N(D)$ has degree 0, so if $g \in E$ then $|g| \in K$. Consider the map $g \to |g|$ from $E$ to $K$ and denote its image by $|E|$. This map is clearly an $F$-monomorphism and we merely have to prove that it is onto.

If $0 \neq u \in R_f$ then $\mathrm{ad}_f^k u = 0$, $\mathrm{ad}_f^{k-1} u \neq 0$ for some $k \geq 1$. It follows that $\mathrm{ad}_f^{k-1} u \in C(f) = E$ so $\deg(\mathrm{ad}_f^{k-1} u) = 0$. Since $\deg f = 0$ it follows that $\deg u = n \geq 0$. If $|u| = u_n x^n$ then $\{f, u\} = -nu_n \delta(|f|)x^{n-1}$ and $\delta(|f|) \neq 0$ since $|f| \notin F$. Applying $\mathrm{Ad}_f$ again, we get $\mathrm{Ad}_f^2 u = n(n-1)u_n \delta(|f|)^2 x^{n-2}$ and, after $n$ steps, $\mathrm{Ad}_f^n u = (-1)^n n! u_n \delta(|f|)^n \in K$. It follows that $|\mathrm{ad}_f^n u| = \mathrm{Ad}_f^n u$ and $k-1 \geq n$. Since $\deg(\mathrm{ad}_f^{k-1} u) = 0 = \deg(\mathrm{ad}_f^n u)$ we deduce $k-1 = n$, so $\mathrm{ad}_f^n u \in E$ and therefore $|\mathrm{ad}_f^n u| = (-1)^n n! u_n \delta(|f|)^n \in |E|$.

Now $(R_f) = D$, so if $0 \neq a \in K$ then $a = uv^{-1}$ with $0 \neq u$, $v \in R_f$ and $\deg u = \deg v$ since $\deg a = 0$. Let $|u| = u_n x^n$, $|v| = v_n x^n$ then $(-1)^n n! u_n \delta(|f|)^n \in |E|$ and $(-1)^n n! v_n \delta(|f|)^n \in |E|$, so $u_n v_n^{-1} \in |E|$. But $a = |a| = |uv^{-1}| = u_n v_n^{-1}$ so $a \in |E|$ and the map $g \to |g|$ is onto.

Proof of the Theorem. Let $D_1$ and $D_2$ be $F$-isomorphic. If $f_1 \in N(D_1) \setminus F$ is mapped under the given isomorphism to $f_2$ then $C(f_1)$ is mapped onto $C(f_2)$ and $R_{f_1}$ is mapped onto $R_{f_2}$ so $f_2 \in N(D_2) \setminus F$. By Lemma 5 and the corollary to Lemma 3, $C(f_i)$ is $F$-isomorphic to $K_i$, $i = 1, 2$, and since $C(f_1)$ and $C(f_2)$ are $F$-isomorphic, we get that $K_1$ and $K_2$ are $F$-isomorphic.

We return to one skew field $D = K[x; \delta]$. By Lemmas 1 and 4 and [3, Cor. 4.3] we get that $N(D)$ is a union of maximal subfields of $D$. By Lemma 5 and the corollary to Lemma 3 we have that if $f \in N(D) \setminus F$ then $E = C(f) \simeq K$. If $\mathrm{tr.deg}\, K/F = 1$, then by Lemma 3 and the easy direction of the theorem, this isomorphism extends to an automorphism of $D$. We proceed to show that the same result holds if $\mathrm{tr.deg}\, K/F > 1$. Choose an element $y \in R_f^1 \setminus R_f^0$. As was shown in the proof of Lemma 5, $|y| = y_1 x$ where $y_1 \in K$. Since $|E| = K$, there exists an element $e \in E$ such that $|e| = y_1$. Let us replace $y$ by $z = e^{-1}y$, then $|z| = x$ and it is easy to see, as in the proof of Lemma 3, that the skew field of quotients of

$E[z]$ is $D$. Now the map $g \to |g|$, $g \in E$ and $z \to x$ gives an isomorphism between $E[z]$ and $K[x]$ which extends to an automorphism of $D$. Thus we have:

COROLLARY. *$N(D)$ is a union of maximal subfields of $D$ each being isomorphic to $K$ and these isomorphisms may be extended to automorphisms of $D$.*

REMARK. A. Schofield showed in [8] that if tr.deg$K/F > 1$, then any subfield $E$ of $D$ with tr.deg$E/F > 1$ is conjugate to a subfield of $K$. From this it follows immediately that, when tr.deg$K/F > 1$, the field $K$ is determined by $D$ and all the automorphisms in the previous corollary may be chosen to be inner.

## REFERENCES

1. S. A. Amitsur, *Commutative linear differential operators*, Pacific J. Math. 8 (1958), 1–10.

2. P. M. Cohn, *Skew Field Constructions*, London Math. Soc. Lecture Note Series **27**, Cambridge University Press, 1977.

3. J. Dixmier, *Sur les algèbres de Weyl*, Bull. Soc. Math. France **96** (1968), 209–242.

4. A. Grothendieck, *Eléments de géométrie algébrique IV*, IHES Publ. Math. **32** (1967).

5. I. N. Herstein, *Noncommutative Rings*, Carus Mathematical Monograph 15, Wiley, New York, 1968.

6. N. Jacobson, *Basic Algebra II*, Freeman, San Francisco, 1980.

7. J. C. McConell and J. C. Robson, *Noncommutative Noetherian Rings*, Wiley, New York, 1987.

8. A. Schofield, *Rigid division rings*, in *Ring Theory 1989*, Israel Mathematical Conference Proceedings, Vol. 1, Weizmann Science Press of Israel, Jerusalem, 1989, pp. 389–394.

9. S. P. Smith and J. T. Stafford, *Differential Opeators on an Affine Curve*, Proc. London Math. Soc. **56** (1988), 229–259.